$F(R)$ bigravity

Shin'ichi Nojiri
Department of Physics &
Kobayashi-Maskawa Institute
for the Origin of Particles and the Universe (KMI),
Nagoya Univ.

“bigravity”
$=$ system of massive spin 2 field (massive graviton)
$+$ gravity (includes massless spin 2 field $=$ graviton)

$F(R)$ extension of bigravity

Application to Cosmology

Contents

Introduction

Mainly based on

S. Nojiri and S. D. Odintsov, “Ghost-free $F(R)$ bigravity and accelerating cosmology,”
Phys. Lett. B **716**, 377 (2012) [arXiv:1207.5106 [hep-th]].
and
S. Nojiri, S. D. Odintsov, and N. Shirai, “Variety of cosmic acceleration models from massive $F(R)$ bigravity,”
JCAP **1305** (2013) 020

♠ Motivation or status of the massive gravity:

- Consistent interacting theory of massive spin-2 field?
  cf. String Theory, Kaluza-Klein Theory.

- Fierz-Pauli action (linearized or free theory)
  M. Fierz and W. Pauli, “On relativistic wave equations for particles of arbitrary spin in an electromagnetic field,” Proc. Roy. Soc. Lond. A **173** (1939) 211.

  The Lagrangian of the massless spin-two field (graviton) $h_{\mu\nu}$ is given by $\left(h \equiv h^{\mu}{}_{\mu}\right)$.

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\lambda h^{\lambda}{}_{\mu}\partial_\nu h^{\mu\nu} - \partial^\mu h_{\mu\nu}\partial^\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \, .$$

  Massive graviton: 5 degrees of freedom.

The Lagrangian of the massive graviton with mass m is given by

$$\mathcal{L}_m = \mathcal{L}_0 - \frac{m^2}{2}\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) .$$

- Boulware-Deser ghost.

  D. G. Boulware and S. Deser, “Classical General Relativity Derived from Quantum Gravity,” Annals Phys. **89** (1975) 193.

  In non-linear (interacting) theory, 6th degree of freedom appears as a ghost.

Massive gravity without ghost

C. de Rham and G. Gabadadze, “Generalization of the Fierz-Pauli Action,” Phys. Rev. D **82**, 044020 (2010) [arXiv:1007.0443 [hep-th]].

C. de Rham, G. Gabadadze and A. J. Tolley, “Resummation of Massive Gravity,” Phys. Rev. Lett. **106** (2011) 231101 [arXiv:1011.1232 [hep-th]].

S. F. Hassan and R. A. Rosen, “Resolving the Ghost Problem in non-Linear Massive Gravity,” Phys. Rev. Lett. **108** (2012) 041101 [arXiv:1106.3344 [hep-th]].

Non-dynamical metric $f_{\mu\nu}\,(\sim\eta_{\mu\nu})$

$$\sqrt{g^{-1}f}\,:\ \sqrt{g^{-1}f}\sqrt{g^{-1}f}=g^{\mu\lambda}f_{\lambda\nu}\,.$$

Minimal extension of Fierz-Pauli action:

$$S=M_p^2\int d^4x\sqrt{-g}\,\left[R-2m^2\left(\mathrm{tr}\sqrt{g^{-1}f}-3\right)\right]\,.$$

$$S = M_p^2 \int d^4x \sqrt{-g} \left[R + 2m^2 \sum_{n=0}^{3} \beta_n \, e_n(\sqrt{g^{-1}f}) \right] \, ,$$

$$e_0(\mathbb{X}) = 1 \, , \quad e_1(\mathbb{X}) = [\mathbb{X}] \, , \quad e_2(\mathbb{X}) = \tfrac{1}{2}([\mathbb{X}]^2 - [\mathbb{X}^2]) \, ,$$

$$e_3(\mathbb{X}) = \tfrac{1}{6}([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3]) \, ,$$

$$e_4(\mathbb{X}) = \tfrac{1}{24}([\mathbb{X}]^4 - 6[\mathbb{X}]^2[\mathbb{X}^2] + 3[\mathbb{X}^2]^2 + 8[\mathbb{X}][\mathbb{X}^3] - 6[\mathbb{X}^4]) \, ,$$

$$e_k(\mathbb{X}) = 0 \ \text{ for } k > 4 \, ,$$

$$\mathbb{X} = (X^\mu{}_\nu) \ , \quad [\mathbb{X}] \equiv X^\mu{}_\mu \, ,$$

$\sim$ Galileon

Bimetric gravity

S. F. Hassan and R. A. Rosen, “Bimetric Gravity from Ghost-free Massive Gravity,” JHEP **1202** (2012) 126 [arXiv:1109.3515 [hep-th]].

Dynamical $f_{\mu\nu}$ (background independent).

$$
\begin{aligned}
S =& M_g^2 \int d^4x \sqrt{-\det g}\, R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f}\, R^{(f)} \\
&+2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n\, e_n(\sqrt{g^{-1}f})\,, \\
&1/M_{\mathrm{eff}}^2 \equiv 1/M_g^2 + 1/M_f^2\,.
\end{aligned}
$$

$R^{(g)}$: scalar curvature for $g_{\mu\nu}$,
$R^{(f)}$: scalar curvature for $f_{\mu\nu}$.

Spectrum of the linearized theory

Minimal case:

$$\beta_0 = 3\,, \quad \beta_1 = -1\,, \quad \beta_2 = 0\,, \quad \beta_3 = 0\,, \quad \beta_4 = 1\,.$$

Linearize

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_g} h_{\mu\nu}\,, \quad f_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_f} l_{\mu\nu}\,.$$

$\Rightarrow$

$$S = \int d^4x \, (h_{\mu\nu}\hat{\mathcal{E}}^{\mu\nu\alpha\beta}h_{\alpha\beta} + l_{\mu\nu}\hat{\mathcal{E}}^{\mu\nu\alpha\beta}l_{\alpha\beta})$$
$$-\frac{m^2 M_{\rm eff}^2}{4}\int d^4x \left[\left(\frac{h^\mu_{\ \nu}}{M_g} - \frac{l^\mu_{\ \nu}}{M_f}\right)^2 - \left(\frac{h^\mu_{\ \mu}}{M_g} - \frac{l^\mu_{\ \mu}}{M_f}\right)^2\right].$$

$\hat{\mathcal{E}}^{\mu\nu\alpha\beta}$: usual Einstein-Hilbert kinetic operator.

Change of variables

$$\frac{1}{M_{\mathrm{eff}}} u_{\mu\nu} = \frac{1}{M_f} h_{\mu\nu} + \frac{1}{M_g} l_{\mu\nu} ,$$
$$\frac{1}{M_{\mathrm{eff}}} v_{\mu\nu} = \frac{1}{M_g} h_{\mu\nu} - \frac{1}{M_f} l_{\mu\nu} .$$

$\Rightarrow$

$$
\begin{aligned}
S &= \int d^4x \left(u_{\mu\nu}\hat{\mathcal{E}}^{\mu\nu\alpha\beta}u_{\alpha\beta} + v_{\mu\nu}\hat{\mathcal{E}}^{\mu\nu\alpha\beta}v_{\alpha\beta}\right) \\
&\quad -\frac{m^2}{4}\int d^4x \, \left(v^{\mu\nu}v_{\mu\nu} - v^{\mu}_{\ \mu}v^{\nu}_{\ \nu}\right) \, .
\end{aligned}
$$

One massless spin-2 particle $u_{\mu\nu}$ and one massive spin-2 particle $v_{\mu\nu}$ with mass m.

Standard $F(R)$ gravity $\Leftrightarrow$ scalar tensor theory

$$S_{F(R)} = \int d^4x\sqrt{-g}\left(\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\mathrm{matter}}\right) .$$

Introducing the auxiliary field A,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{F'(A)\left(R-A\right) + F(A)\right\} .$$

Variation of $A \Rightarrow A = R$: original action

Rescaling of metric

$$g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu} \, , \quad \sigma = -\ln F'(A) \, .$$

$\Rightarrow$ Einstein frame action:

$$S_E = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) \right) \, ,$$

$$V(\sigma) = \mathrm{e}^{\sigma} g\left(\mathrm{e}^{-\sigma}\right) - \mathrm{e}^{2\sigma} f\left(g\left(\mathrm{e}^{-\sigma}\right)\right) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \, .$$

$$A = g\left(\mathrm{e}^{-\sigma}\right) \Leftarrow \sigma = -\ln\left(1 + f'(A)\right) = -\ln F'(A)$$

Coupling of σ with matters appears by the rescaling $g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu}$.

Adding the following actions to the bigravity action

$$S_\varphi = -\, M_g^2 \int d^4x \sqrt{-\det g} \left\{ \frac{3}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \right\}$$
$$+ \int d^4x \mathcal{L}_{\mathrm{matter}} \left(\mathrm{e}^{\varphi} g_{\mu\nu}, \Phi_i \right) ,$$
$$S_\xi = -\, M_f^2 \int d^4x \sqrt{-\det f} \left\{ \frac{3}{2} f^{\mu\nu} \partial_\mu \xi \partial_\nu \xi + U(\xi) \right\} .$$

Conformal transformations

$$g_{\mu\nu} \to \mathrm{e}^{-\varphi} g_{\mu\nu} \, , \quad f_{\mu\nu} \to \mathrm{e}^{-\xi} f_{\mu\nu} \, ,$$

$$
\begin{aligned}
S_F =& M_f^2 \int d^4x \sqrt{-\det f^{\mathrm{J}}}\, \left\{ \mathrm{e}^{-\xi} R^{\mathrm{J}(f)} - \mathrm{e}^{-2\xi} U(\xi) \right\} \\
&+ 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}} \sum_{n=0}^{4} \beta_n \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \\
&+ M_g^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}} \left\{ \mathrm{e}^{-\varphi} R^{\mathrm{J}(g)} - \mathrm{e}^{-2\varphi} V(\varphi) \right\} \\
&+ \int d^4x \mathcal{L}_{\mathrm{matter}} \left(g^{\mathrm{J}}_{\mu\nu}, \Phi_i \right) \ .
\end{aligned}
$$

Kinetic terms of φ and ξ vanish.

Coupling of φ with matters also disappears.

Variation of φ and ξ $\Rightarrow$

$$
\begin{aligned}
0 =& 2m^2 M_{\mathrm{eff}}^2 \sum_{n=0}^{4} \beta_n \left(\frac{n}{2} - 2\right) \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right) \\
& + M_g^2 \left\{ -\mathrm{e}^{-\varphi} R^{\mathrm{J}(g)} + 2\mathrm{e}^{-2\varphi} V(\varphi) + \mathrm{e}^{-2\varphi} V'(\varphi) \right\} \, , \\
0 =& - 2m^2 M_{\mathrm{eff}}^2 \sum_{n=0}^{4} \frac{\beta_n n}{2} \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right) \\
& + M_f^2 \left\{ -\mathrm{e}^{-\xi} R^{\mathrm{J}(f)} + 2\mathrm{e}^{-2\xi} U(\xi) + \mathrm{e}^{-2\xi} U'(\xi) \right\} \, .
\end{aligned}
$$

In principle, can be solved algebraically with respect to φ and ξ

$$
\varphi = \varphi \left(R^{(g)}, R^{(f)}, e_n \left(\sqrt{g^{-1} f} \right) \right) \, , \quad \xi = \xi \left(R^{(g)}, R^{(f)}, e_n \left(\sqrt{g^{-1} f} \right) \right) \, .
$$

$\Rightarrow$ analogue of $F(R)$ gravity:

$$
\begin{aligned}
S_F =& M_f^2 \int d^4x \sqrt{-\det f^{\mathrm{J}}}\, F^{\mathrm{J}(f)} \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \right) \\
&+ 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi\left(R^{\mathrm{J}(g)}, e_n\left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}}\right)\right)} e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \\
&+ M_g^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}}\, F^{\mathrm{J}(g)} \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \right) \\
&+ \int d^4x \mathcal{L}_{\mathrm{matter}} \left(g^{\mathrm{J}}_{\mu\nu}, \Phi_i \right) \,,
\end{aligned}
$$

Here

$$F^{\mathrm{J}(g)}\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right) \equiv \left\{ \mathrm{e}^{-\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)} R^{\mathrm{J}(g)} \right.$$
$$\left. - \mathrm{e}^{-2\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)} V\left(\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)\right)\right\},$$
$$F^{\mathrm{J}(f)}\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right) \equiv \left\{ \mathrm{e}^{-\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)} R^{\mathrm{J}(f)} \right.$$
$$\left. - \mathrm{e}^{-2\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)} U\left(\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)\right)\right\}.$$

It is difficult to explicitly solve equations with respect to φ and ξ and it might be better to define the model by introducing the auxiliary scalar fields φ and ξ.

Cosmological Reconstruction

Minimal case:

$$S_{\mathrm{bi}} = M_g^2 \int d^4x \sqrt{-\det g}\, R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f}\, R^{(f)} + 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \left(3 - \mathrm{tr}\, \sqrt{g^{-1} f} + \det \sqrt{g^{-1} f}\right) .$$

Evaluation of $\delta\sqrt{g^{-1}f}$:

Two matrices M, N: $M^2 = N$.

$$\delta M M + M \delta M = \delta N \ \Rightarrow\ \mathrm{tr}\, \delta M = \frac{1}{2} \mathrm{tr}\, \left(M^{-1} \delta N\right) .$$

Start from the Einstein frame action. Neglect matter.

$\delta g_{\mu\nu} \ \Rightarrow$

$$0 = M_g^2 \left(\frac{1}{2} g_{\mu\nu} R^{(g)} - R^{(g)}_{\mu\nu} \right)$$
$$+ m^2 M_{\text{eff}}^2 \left\{ g_{\mu\nu} \left(3 - \mathrm{tr}\, \sqrt{g^{-1} f} \right) + \frac{1}{2} f_{\mu\rho} \left(\sqrt{g^{-1} f} \right)^{-1\,\rho}_{\nu} + \frac{1}{2} f_{\nu\rho} \left(\sqrt{g^{-1} f} \right)^{-1\,\rho}_{\mu} \right\}$$
$$+ M_g^2 \left[\frac{1}{2} \left(\frac{3}{2} g^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi + V(\varphi) \right) g_{\mu\nu} - \frac{3}{2} \partial_\mu \varphi \partial_\nu \varphi \right] .$$

$\delta f_{\mu\nu} \ \Rightarrow$

$$0 = M_f^2 \left(\frac{1}{2} f_{\mu\nu} R^{(f)} - R^{(f)}_{\mu\nu} \right)$$
$$+ m^2 M_{\text{eff}}^2 \sqrt{\det\ (f^{-1} g)} \left\{ - \frac{1}{2} f_{\mu\rho} \left(\sqrt{g^{-1} f} \right)^{\rho}_{\ \nu} - \frac{1}{2} f_{\nu\rho} \left(\sqrt{g^{-1} f} \right)^{\rho}_{\ \mu} + \det\ \left(\sqrt{g^{-1} f} \right) f_{\mu\nu} \right\}$$
$$+ M_f^2 \left[\frac{1}{2} \left(\frac{3}{2} f^{\rho\sigma} \partial_\rho \xi \partial_\sigma \xi + U(\xi) \right) f_{\mu\nu} - \frac{3}{2} \partial_\mu \xi \partial_\nu \xi \right] .$$

$\delta\varphi$, $\delta\xi \Rightarrow$

$$0 = -3\Box_g \varphi + V'(\varphi)\,, \quad 0 = -3\Box_f \xi + U'(\xi)\,.$$

$\Box_g$, $\Box_f$: d'Alembertian w.r.t. g, f.
Bianchi identity $0 = \nabla_g^\mu \left(\frac{1}{2} g_{\mu\nu} R^{(g)} - R^{(g)}_{\mu\nu} \right)$ + Einstein like equations $\Rightarrow$

$$\begin{aligned} 0 =& - g_{\mu\nu} \nabla_g^\mu \left(\mathrm{tr}\, \sqrt{g^{-1} f} \right) \\ & + \frac{1}{2} \nabla_g^\mu \left\{ f_{\mu\rho} \left(\sqrt{g^{-1} f} \right)^{-1\,\rho}_{\quad\ \ \nu} + f_{\nu\rho} \left(\sqrt{g^{-1} f} \right)^{-1\,\rho}_{\quad\ \ \mu} \right\}\,. \end{aligned}$$

Similarly

$$
\begin{aligned}
0 = & \nabla_f^{\mu} \left[\sqrt{\det \left(f^{-1} g \right)} \left\{ -\frac{1}{2} \left(\sqrt{g^{-1} f} \right)^{-1\nu}{}_{\sigma} g^{\sigma\mu} \right. \right. \\
& \left. \left. -\frac{1}{2} \left(\sqrt{g^{-1} f} \right)^{-1\mu}{}_{\sigma} g^{\sigma\nu} + \det \left(\sqrt{g^{-1} f} \right) f^{\mu\nu} \right\} \right] .
\end{aligned}
$$

In case of the Einstein gravity,

conservation law $\Leftarrow$ Einstein equation $+$ Bianchi identity

or conservation laws $\Leftarrow$ scalar field equations

In case of bigravity, only

conservation laws $\Leftarrow$ scalar field equations

Bianchi identities

$\Rightarrow$ new equations independent of Einstein like equation.

Assume FRW universes by using the conformal time t

$$ds_g^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu = a(t)^2 \left(-dt^2 + \sum_{i=1}^{3} \left(dx^i\right)^2 \right) ,$$

$$ds_f^2 = \sum_{\mu,\nu=0}^{3} f_{\mu\nu} dx^\mu dx^\nu = -c(t)^2 dt^2 + b(t)^2 \sum_{i=1}^{3} \left(dx^i\right)^2 .$$

$\Rightarrow$

$$\delta g_{tt} \ : \ 0 = -\, 3 M_g^2 H^2 - 3 m^2 M_{\text{eff}}^2 \left(a^2 - ab\right) + \left(\frac{3}{4} \dot{\varphi}^2 + \frac{1}{2} V(\varphi) a(t)^2 \right) M_g^2 ,$$

$$\delta g_{ij} \ : \ 0 = M_g^2 \left(2\dot{H} + H^2\right) + m^2 M_{\text{eff}}^2 \left(3a^2 - 2ab - ac\right) + \left(\frac{3}{4} \dot{\varphi}^2 - \frac{1}{2} V(\varphi) a(t)^2 \right) M_g^2 , \qquad H \equiv \frac{\dot{a}}{a} .$$

$$\delta f_{tt} \ : \ 0 = -3M_f^2 K^2 + m^2 M_{\rm eff}^2 c^2 \left(1 - \frac{a^3}{b^3}\right) + \left(\frac{3}{4}\dot\xi^2 - \frac{1}{2}U(\xi)c(t)^2\right) M_f^2 \,,$$

$$\delta f_{ij} \ : \ 0 = M_f^2 \left(2\dot K + 3K^2 - 2LK\right) + m^2 M_{\rm eff}^2 \left(\frac{a^3 c}{b^2} - c^2\right) + \left(\frac{3}{4}\dot\xi^2 - \frac{1}{2}U(\xi)c(t)^2\right) M_f^2 \,.$$

$$K \equiv \dot b / b \,, \quad L = \dot c / c \,.$$

Both of equations derived from Bianchi identity:

$cH = bK$ or $\frac{c\dot a}{a} = \dot b$.

If $\dot a \neq 0$, we obtain $c = a\dot b/\dot a$.

If $\dot a = 0$, we find $\dot b = 0$, that is, a, b: constant, c can be arbitrary.

Redefinition of the scalar fields: $\varphi = \varphi(\eta)$, $\xi = \xi(\zeta)$.
Identify $\eta = \zeta = t$

$$\begin{aligned}
\omega(t) M_g^2 &= -\,4M_g^2\left(\dot{H} - H^2\right) - 2m^2 M_{\mathrm{eff}}^2 (ab - ac)\,,\\
\tilde{V}(t) a(t)^2 M_g^2 &= M_g^2\left(2\dot{H} + 4H^2\right) + m^2 M_{\mathrm{eff}}^2 (6a^2 - 5ab - ac)\,,\\
\sigma(t) M_f^2 &= -\,4M_f^2\left(\dot{K} - LK\right) - 2m^2 M_{\mathrm{eff}}^2 \left(-\frac{c}{b} + 1\right)\frac{a^3 c}{b^2}\,,\\
\tilde{U}(t) c(t)^2 M_f^2 &= M_f^2\left(2\dot{K} + 6K^2 - 2LK\right) + m^2 M_{\mathrm{eff}}^2 \left(\frac{a^3 c}{b^2} - 2c^2 + \frac{a^3 c^2}{b^3}\right)\,.
\end{aligned}$$

$$\omega(\eta) = 3\varphi'(\eta)^2,\ \tilde{V}(\eta) = V\left(\varphi\left(\eta\right)\right),\ \sigma(\zeta) = 3\xi'(\zeta)^2,\ \tilde{U}(\zeta) = U\left(\xi\left(\zeta\right)\right)\,.$$

For arbitrary $a(t)$ and $b(t)$, if we choose $\omega(t)$, $\tilde{V}(t)$, $\sigma(t)$, and $\tilde{U}(t)$ to satisfy the above equations, a model admitting the given $a(t)$ and $b(t)$ evolution can be reconstructed.

Physical metric: the scalar field does not directly coupled with matter: $g^{\mathrm{J}}_{\mu\nu} = \mathrm{e}^{\varphi} g_{\mu\nu}$.

$$\text{FRW universe:} \quad ds^2 = \tilde{a}(t)^2 \left(-dt^2 + \sum_{i=1}^{3} \left(dx^i\right)^2 \right) .$$

$\tilde{a}(t)^2 = \frac{l^2}{t^2}$: de Sitter universe.
$\tilde{a}(t)^2 = \frac{l^{2n}}{t^{2n}}$ with $n \neq 1$ case:
Redefinition of time coordinate: $d\tilde{t} = \pm \frac{l^n}{t^n} dt \ \left(\tilde{t} = \pm \frac{l^n}{n-1} t^{1-n} \right)$

$$\Rightarrow \ ds^2 = -d\tilde{t}^2 + \left(\pm (n-1) \frac{\tilde{t}}{l} \right)^{-\frac{2n}{1-n}} \sum_{i=1}^{3} \left(dx^i\right)^2 .$$

$0 < n < 1$: phantom universe, $n > 1$: quintessence universe, $n < 0$: decelerating universe

Universe with $a(t) = b(t) = 1$

$a(t) = b(t) = 1$ satisfies the previous constraint.
$\Rightarrow$ Einstein frame metric $g_{\mu\nu}$: flat Minkowski space metric we observe: $g^{\rm J}_{\mu\nu}$.

$$
\begin{aligned}
\omega(t)^2 M_g^2 =& 12 M_g^2 \tilde H^2 = m^2 M_{\rm eff}^2 \left(c - 1\right) \, , \\
\tilde V(t) M_g^2 =& m^2 M_{\rm eff}^2 \left(1 - c\right) = - 6 M_g^2 \tilde H^2 \, , \\
\sigma(t) M_f^2 =& 2 m^2 M_{\rm eff}^2 \left(c - 1\right) = 12 M_g^2 \tilde H^2 \, , \\
\tilde U(t) M_f^2 =& m^2 M_{\rm eff}^2 c \left(1 - c\right) = - 6 M_g^2 \tilde H^2 \left(1 + \frac{6 \tilde H^2}{m^2 M_{\rm eff}^2} \right) \, ,
\end{aligned}
$$

$$
\Rightarrow \quad c = 1 + \frac{6 \tilde H^2}{m^2 M_{\rm eff}^2} \, .
$$

Note: $\omega(t)$, $\sigma(t) > 0$ (no ghost)

Big Rip, quintessence, de Sitter and decelerating universes

$\tilde{a}(t)^2 = \frac{l^{2n}}{t^{2n}}$

$$\omega(t)^2 M_g^2 = \frac{12n^2 M_g^2}{t^2}\,, \quad \tilde{V}(t) M_g^2 = -\frac{6n^2 M_g^2}{t^2}\,,$$

$$\sigma(t) M_f^2 = \frac{12n^2 M_g^2}{t^2}\,, \quad \tilde{U}(t) M_f^2 = -\frac{6n^2 M_g^2}{t^2}\left(1 + \frac{6n^2}{m^2 M_{\mathrm{eff}}^2 t^2}\right)\,.$$

$$\Rightarrow \quad \mathrm{e}^{\xi} = \frac{n^2}{t^2}\,,$$

$$\left(ds_f^{\mathrm{J}}\right)^2 = \sum_{\mu,\nu=0}^{3} f_{\mu\nu}^{\mathrm{J}} dx^\mu dx^\nu$$

$$= \mathrm{e}^{\xi} ds_f^2 = \frac{n^2}{t^2}\left\{ - \left(1 + \frac{6n^2}{m^2 M_{\mathrm{eff}}^2 t^2}\right)^2 dt^2 + \left(dx^i\right)^2 \right\}\,.$$

When $t \sim 0$, redefinition:

$$\tilde{t} \sim \frac{\alpha}{2t^2}\,, \quad \alpha \equiv \frac{6n^3}{m^2 M_{\mathrm{eff}}^2 t^2}\,,$$

$\Rightarrow$

$$\left(ds_f^{\mathrm{J}}\right)^2 \sim -d\tilde{t}^2 + \frac{2n^2\tilde{t}}{\alpha}\left(dx^i\right)^2\,.$$

$t \to 0 \Leftrightarrow \tilde{t} \to +\infty$.
There does not occur singularity in the metric $\left(ds_f^{\mathrm{J}}\right)^2$ because the scale factor $\tilde{a}$ which is proportional to $\tilde{t}$ corresponds to the universe filled with radiation.

Super-luminal mode in bigravity

There can be a signal whose speed is larger than the speed of light.

Speed v_g of the massless particle which propagates in the universe described by $g^{\mathrm{J}}_{\mu\nu}$ or $g_{\mu\nu}$

$$v_g^2 = (dx/dt)^2 = 1 \ \Leftrightarrow \ \text{special relativity.}$$

Speed v_f in $f^{\mathrm{J}}_{\mu\nu}$ or $f_{\mu\nu}$

$$v_f^2 = (dx/dt)^2 = c(t)^2/b(t)^2$$

If $c(t)/b(t) > 1$, $v_f > 0$speed of light in g universe.
$c(t) > 1$ except of $\tilde{H} = 0$: $v_f = 1 + \frac{6\tilde{H}^2}{m^2 M_{\mathrm{eff}}^2} > 1$.
v_f is greater than the speed of light.

Summary

- $F(R)$ bigravity in the conventional description with two metrics g and f.
- Explicit and exact solution of FRW equations, Big (and Little) Rip, de Sitter, quintessence and decelerating universes.
- In general, the physical g cosmological singularity is manifested as metric f cosmological singularity. However, there are examples where cosmological singularity of physical g universe does not occur in the universe described by reference metric f and vice-versa.
- The massless particle in the space-time given by the metric $f_{\mu\nu}$ or $f^{\mathrm{J}}_{\mu\nu}$ can be super-luminal.
- Other models, scalar-tensor, Brans-Dicke.